

Quasi-spherical collapse with cosmological constant

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The junction conditions between static and non-static space-times are studied for analyzing gravitational collapse in the presence of a cosmological constant. We have discussed about the apparent horizon and their physical significance. We also show the effect of cosmological constant in the collapse and it has been shown that cosmological constant slows down the collapse of matter.

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I. INTRODUCTION

Gravitational collapse is one of the most important problem in classical general relativity. Usually, the formation of compact stellar objects such as white dwarf and neutron star are preceded by a period of collapse. Hence for astrophysical collapse, it is necessary to describe the appropriate geometry of interior and exterior regions and to determine proper junction conditions which allow the matching of these regions.

The study of gravitational collapse was started by Oppenheimer and Snyder [1]. They studied collapse of dust with a static Schwarzschild exterior while interior space-time is represented by Friedman like solution. Since then several authors have extended the above study of collapse of which important and realistic generalizations are the following: (i) the static exterior was studied by Misner and Sharp [2] for a perfect fluid in the interior, (ii) using the idea of outgoing radiation of the collapsing body by Vaidya [3], Santos and collaborations [4-9] included the dissipation in the source by allowing radial heat flow (while the body undergoes radiating collapse). Ghosh and Deskar [10] have considered collapse of a radiating star with a plane symmetric boundary (which has a close resemblance with spherical symmetry [11]) and have concluded with some general remarks. On the othehand, Cissoko et al [12] have studied junction conditions between static and non-static space-times for analyzing gravitational collapse in the presence of dark energy has been investigated by Mota et al [13] and Cai et al [14]. The effect of cosmological constant (a source of dark energy) in cosmology has been shown by Lahav et al [15] and Antolinez et al [16].

So far most of the studies have considered in a star whose interior geometry is spherical. But in the real astrophysical situation the geometry of the interior of a star may not be exactly spherical, rather quasi-spherical in form. Recently, solutions for arbitrary dimensional Szekeres' model with perfect fluid (or dust) [17] has been found for quasi-spherical or quasi-cylindrical symmetry of the space-time. Also a detailed analysis of the gravitational collapse [18, 19] has been done for quasi-spherical symmetry of the Szekeres' model. It has also been studied junction conditions between quasi-spherical interior geometry of radiating star and exterior Vaidya metric [20]. In this paper, we have considered the interior space-time V^- by Szekeres' model [17, 21] while for exterior geometry V^+ we have considered Schwarzschild-de-Sitter space-time. The plan of the paper is as follows: The junction conditions has been presented in section II. The apparent horizons and their physical interpretations are shown in section III. The paper ends with a short conclusion in section IV.

II. JUNCTION CONDITIONS

Let us consider a time-like $3D$ hypersurface Σ , which divides $4D$ space-time into two distinct $4D$ manifolds V^- and V^+ . For junction conditions we follow the modified version of Israel [22] by Santos [4, 5]. Now the

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geometry of the space-time V^- inside the boundary Σ is given by the Szekeres space-time

$$ds_-^2 = -dt^2 + e^{2\alpha}dr^2 + e^{2\beta}(dx^2 + dy^2) \quad (1)$$

where α and β are functions of all space-time variables.

The metric co-efficients α and β have the explicit form for dust matter with cosmological constant Λ [13, 16]

$$e^\beta = R(t, r)e^{\nu(r, x, y)} \quad (2)$$

$$e^\alpha = R' + R\nu' \quad (3)$$

The evolution equation for R is

$$\dot{R}^2 = f(r) + \frac{F(r)}{R} + \frac{\Lambda}{3} R^2 \quad (4)$$

and ν has the explicit form

$$e^{-\nu} = A(r)(x^2 + y^2) + B_1(r)x + B_2(r)y + C(r) \quad (5)$$

where $F(r)$ (> 0) and $f(r)$ are arbitrary functions of r and $A(r)$, $B_1(r)$, $B_2(r)$ and $C(r)$ are arbitrary functions of r along with the restriction

$$B_1^2 + B_2^2 - 4AC = f(r) - 1 \quad (6)$$

Assuming $x = \cot(\theta/2)\cos\phi$, $y = \cot(\theta/2)\sin\phi$, equation (1) becomes

$$ds_-^2 = -dt^2 + e^{2\alpha}dr^2 + \frac{1}{4} e^{4\beta} \text{cosec}^4(\theta/2)(d\theta^2 + \sin^2\theta d\phi^2) \quad (7)$$

For exterior space-time V^+ to Σ , we have considered the Schaezschild-de-Sitter space-time

$$ds_+^2 = -N(z)dT^2 + \frac{1}{N(z)}dz^2 + (d\theta^2 + \sin^2\theta d\phi^2) \quad (8)$$

where $N(z) = (1 - \frac{2M}{z} - \frac{\Lambda}{3} z^2)$, M is a constant.

The intrinsic metric on the boundary Σ of the hypersurface $r = r_\Sigma$ is given by

$$ds_\Sigma^2 = -d\tau^2 + A^2(\tau)(d\theta^2 + \sin^2\theta d\phi^2) \quad (9)$$

Now Israel's junction conditions (as described by Santos [4, 5]) are

(i) The continuity of the line element i.e.,

$$(ds_-^2)_\Sigma = (ds_+^2)_\Sigma = ds_\Sigma^2 \quad (10)$$

where $()_\Sigma$ means the value of $()$ on Σ .

(ii) The continuity of extrinsic curvature over Σ gives

$$[K_{ij}] = K_{ij}^+ - K_{ij}^- = 0, \quad (11)$$

where due to Eisenhart [23] the extrinsic curvature has the expresion

$$K_{ij}^{\pm} = -n_{\sigma}^{\pm} \frac{\partial^2 \chi_{\pm}^{\sigma}}{\partial \xi^i \partial \xi^j} - n_{\sigma}^{\pm} \Gamma_{\mu\nu}^{\sigma} \frac{\partial \chi_{\pm}^{\mu}}{\partial \xi^i} \frac{\partial \xi_{\pm}^{\nu}}{\partial \xi^j} \quad (12)$$

Here $\xi^i = (\tau, x, y)$ are the intrinsic co-ordinates to Σ , χ_{\pm}^{σ} , $\sigma = 0, 1, 2, 3$ are the co-ordinates in V^{\pm} and n_{α}^{\pm} are the components of the normal vector to Σ in the co-ordinates χ_{\pm}^{σ} .

Now for the interior space-time described by the metric (1) the boundary of the interior matter distribution (i.e., the surface Σ) is characterized by

$$f(r, t) = r - r_{\Sigma} = 0 \quad (13)$$

where r_{Σ} is a constant. As the vector with components $\frac{\partial f}{\partial \chi_{\pm}^{\sigma}}$ is orthogonal to Σ so we take

$$n_{\mu}^{-} = (0, e^{\alpha}, 0, 0).$$

So comparing the metric ansatzs given by equations (1) and (9) for $dr = 0$ we have from the continuity relation (10)

$$\frac{dt}{d\tau} = 1, \quad A(\tau) = e^{\beta} \quad \text{on} \quad r = r_{\Sigma} \quad (14)$$

Also the components of the extrinsic curvature for the interior space-time are

$$K_{\tau\tau}^{-} = 0 \quad \text{and} \quad K_{\theta\theta}^{-} = \text{cosec}^2 \theta K_{\phi\phi}^{-} = \left[\frac{1}{4} \beta' e^{2\beta-\alpha} \text{cosec}^4(\theta/2) \right]_{\Sigma}. \quad (15)$$

On the other hand for the exterior Schwarzschild-de-Sitter metric described by the equation (8) with its interior boundary, given by

$$f(z, T) = z - z_{\Sigma}(T) = 0 \quad (16)$$

the unit normal vector to Σ is given by

$$n_{\mu}^{+} = \left(N - \frac{1}{N} \left(\frac{dz}{dT} \right)^2 \right)^{-1/2} \left(-\frac{dz}{dT}, 1, 0, 0 \right) \quad (17)$$

and the components of the extrinsic curvature are

$$K_{\tau\tau}^{+} = \left[\frac{N\ddot{T}}{z} + \frac{dN}{dz} \dot{T} \right]_{\Sigma} \quad (18)$$

and

$$K_{\theta\theta}^{+} = \text{cosec}^2 \theta K_{\phi\phi}^{+} = \left[\dot{T} N z \right]_{\Sigma} \quad (19)$$

Hence the continuity of the extrinsic curvature due to junction condition (eq. (11)) gives

$$N = \left[\frac{1}{4} e^{2\beta} \text{cosec}^4(\theta/2) \left(e^{-2\alpha} \beta'^2 - \dot{\beta}^2 \right) \right]_{\Sigma} \quad (20)$$

and

$$\dot{T}_{\Sigma} = \left[2 \sin^2(\theta/2) \left(e^{\beta-\alpha} \beta'^2 - e^{\beta+\alpha} \dot{\beta}^2 \right)^{-1} \right]_{\Sigma} \quad (21)$$

Now using the junction condition (20) with the help of equations (2), (3) and (4), we have (on the boundary) [18]

$$\frac{1}{2}\dot{R}^2 - \frac{M}{R} - \frac{\Lambda}{6} R^2 = 0 \quad (22)$$

which can be interpreted as the energy conservation equation on the boundary. It is to be noted that the cosmological term leads to a repulsive term to the Newtonian potential [16] i.e.,

$$\phi(R) = \frac{M}{R} + \frac{\Lambda}{6} R^2 \quad (23)$$

III. TRAPPED SURFACES : COSMOLOGICAL AND BLACK HOLE HORIZONS

As the present space-time geometry is complicated, so it is difficult to find the formation of event horizon. However, trapped surfaces which are space-like 2-surfaces with normals on both sides are future pointing converging null geodesic families, may be considered here. In fact, if the 2-surface $S_{r,t}$ ($r = \text{constant}$, $t = \text{constant}$) is a trapped surface then it and its entire future development lie behind the event horizon unless the density falls off fast enough at infinity. So if K^μ is the tangent vector field to the null geodesics orthogonal to the trapped surface then K^μ should satisfy (i) $K_\mu K^\mu = 0$, (ii) $K^\mu_{;\nu} K^\nu = 0$.

Also the convergence (or divergence) of the null geodesics on the trapped surface is characterized by the sign of the scalar $K^\mu_{;\mu}$ ($K^\mu_{;\mu} < 0$ for convergence, $K^\mu_{;\mu} > 0$ for divergence). It is to be noted that the inward geodesics converges initially and throughout the collapsing process but the outward geodesics diverges initially but becomes convergent after a time $t_{ah}(r)$ (the time of formation of apparent horizon) given by

$$\dot{R}^2 = 1 + f(r) \quad (24)$$

Then from the evolution equation (4), we have

$$\Lambda R^3 - 3R + 3F(r) = 0 \quad (25)$$

The possible solutions of equation (25) for different choices of Λ and $F(r)$ are shown in the TABLE.

For marginally bound case (i.e., $f(r) = 0$) the evolution equation (4) can be solved as

$$t_c(r) - t = \frac{2}{\sqrt{3\Lambda}} \sinh^{-1} \left[\sqrt{\frac{\Lambda}{3F}} R^{3/2} \right] \quad (26)$$

where, $t = t_c(r)$ is the time of collapse of a shell of radius r (i.e., $R = 0$ at $t = t_c(r)$).

TABLE

Restrictions on Λ , $F(r)$	Solutions of eq.(25): Different horizons
(a) $\Lambda = 0$	$R = F(r)$, Schwarzschild horizon
(c) $F(r) = 0$	$R = 0$ (black hole) $R = \pm \frac{1}{\sqrt{\Lambda}}$ (de-Sitter horizon)
(d) $F(r) < \frac{2}{3} \frac{1}{\sqrt{\Lambda}}$	Two horizons: $R_1 = \frac{2}{\sqrt{\Lambda}} \cos(\theta/3)$ $R_2 = \frac{1}{\sqrt{\Lambda}} [-\cos(\theta/3) + \sqrt{3} \sin(\theta/3)]$ $\cos\theta = -\frac{3}{2} \sqrt{\Lambda} F(r)$ $0 \leq R_2 \leq \sqrt{\Lambda} \leq R_1 \leq \frac{\sqrt{3}}{\sqrt{\Lambda}}$
(e) $F(r) = \frac{2}{3} \frac{1}{\sqrt{\Lambda}}$	$R = 0$ $R = \frac{1}{\sqrt{\Lambda}}$
(f) $F(r) > \frac{2}{3} \frac{1}{\sqrt{\Lambda}}$	no horizon

Hence the time of formation of apparent horizon $t_{ah}(r)$ is given by

$$t_{ah}(r) = t_c(r) - \frac{2}{\sqrt{3\Lambda}} \sinh^{-1} \left[\sqrt{\frac{\Lambda}{3F}} R_H^{3/2} \right] \quad (27)$$

where R_H is a root of the equation (25).

Thus from the above table we see that in the fourth case (i.e., $F(r) < \frac{2}{3} \frac{1}{\sqrt{\Lambda}}$) we have two horizons namely cosmological and black hole horizons ($R_1 \geq R_2$) and let t_1 and t_2 be their time of formation then from equation (27), $t_1 \leq t_2$, i.e., cosmological horizon forms earlier than the formation of black hole horizon.

Further, if T_1 and T_2 be the time differences between the formation of cosmological horizon and singularity and the formation of black hole horizon and singularity respectively then

$$T_i = \frac{2}{\sqrt{3\Lambda}} \sinh^{-1} \left[\sqrt{\frac{\Lambda}{3F}} R_i^{3/2} \right], \quad i = 1, 2. \quad (28)$$

A straight forward calculation shows

$$\frac{dT_1}{dF} < 0 \quad \text{and} \quad \frac{dT_2}{dF} > 0, \quad \frac{dT}{dF} < 0, \quad T = T_1 - T_2. \quad (29)$$

Thus the time difference between the formation of singularity and cosmological horizon decreases with F increases while the time difference between the formation of singularity and black hole horizon increases with F . As F is related to the mass of the collapsing system so for more massive quasi-spherical model, the time of

formation of singularity and cosmological horizon become close to each other while the time difference between the formation of black hole horizon and that of cosmological horizon becomes smaller.

IV. CONCLUSION

In this paper, the collapse of a quasi-spherical star is considered where the exterior geometry corresponds to Schwarzschild-de-Sitter space-time. The junction conditions on the boundary show a energy conservation equation on it.

Due to the presence of the cosmological constant Λ , the Newtonian force is given by (see equation (23)) [24]

$$P(R) = -\frac{M}{R^2} + \frac{\Lambda}{3} R$$

For collapsing process the force should be attractive in nature and as a result R should always be less than $\left(\frac{3M}{\Lambda}\right)^{1/3}$. Further, the rate of collapse has the expression

$$\ddot{R} = -\frac{M}{2R^2} + \frac{\Lambda}{3} R \quad (30)$$

which shows that the presence of Λ -term slows down the collapsing process and hence influences the time difference between the formation of the apparent horizon and the singularity.

As the presence of a cosmological constant (dark energy) induces a potential barrier to the equation of motion so particles with a small velocity are unable to reach the central object. This ideas can be used astrophysically for a particle orbiting a black hole, which contains dark energy and an estimation of minimum velocity can be done for which the particle enters inside the black hole. Consequently, the amount of dark energy in the black hole can be calculated.

Lastly, due to the presence of the cosmological constant, there are two physical horizons – the black hole horizon and the cosmological horizon. Further, for more massive collapsing system, the time of formation of the two horizons become very close to each other. Moreover, asymptotic flatness of the space-time is violated due to the presence of the cosmological constant.

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References:

- [1] J. R. Oppenheimer and H. Snyder, *Phys. Rev.* **56** 455 (1939).
- [2] C. W. Misner and D. Sharp, *Phys. Rev.* **136** b571 (1964).
- [3] P. C. Vaidya, *Proc. Indian Acad. Sci. A* **33** 264 (1951).
- [4] N. O. Santos, *Phys. Lett. A* **106** 296 (1984).
- [5] N. O. Santos, *Mon. Not. R. Astr. Soc.* **216** 403 (1985).
- [6] A. K. G. de Oliveira, N. O. Santos and C. A. Kolassis, *Mon. Not. R. Astr. Soc.* **216** 1001 (1985).
- [7] A. K. G. de Oliveira, J. A. de F. Pacheco and N. O. Santos, *Mon. Not. R. Astr. Soc.* **220** 405 (1986).
- [8] A. K. G. de Oliveira and N. O. Santos, *Astrophys. J.* **312** 640 (1987).
- [9] A. K. G. de Oliveira, C. A. Kolassis and N. O. Santos, *Mon. Not. R. Astr. Soc.* **231** 1011 (1988).
- [10] S. G. Ghosh and D. W. Deshkar, *Int. J. Mod. Phys. D* **12** 317 (2003).
- [11] S. G. Ghosh and D. W. Deshkar, *Gravitation and Cosmology* **6** 1 (2000).
- [12] M. Cissoko, J. Fabris, J. Gariel, G. L. Denmat and N. O. Santos, *gr-qc/9809057*; S. M. C. V. Goncalves, *Class. Quantum Grav.* **18** 4517-4530 (2001).
- [13] D. F. Mota and C. Van de Bruc, *Astron. Astrophys.* **421** 71 (2004).

- [14] R. -G. Cai and A. Wang, *gr-qc/0505136*.
- [15] O. Lahav, P. B. Lilje, J. R. Primack and M. J. Rees, *Mon. Not. R. Astron. Soc.* **128** (1991).
- [16] A. Balaguera-Antolinez, C. G. Boehmer and M. Nowakowski, *Class. Quantum Grav.* **23** 485 (2006).
- [17] S. Chakraborty and U. Debnath, *Int. J. Mod. Phys. D* **13** 1085 (2004); *gr-qc/0304072*.
- [18] P. Szekeres, *Phys. Rev. D* **12** 2941 (1975).
- [19] U. Debnath, S. Chakraborty and J. D. Barrow, *Gen. Rel. Grav.*, **36** 231 (2004); *gr-qc/0305075*.
- [20] U. Debnath, S. Nath and S. Chakraborty, *Gen. Rel. Grav.* **37** 215 (2005).
- [21] P. Szekeres, *Commun. Math. Phys.* **41** 55 (1975).
- [22] W. Israel, *Nuovo Cimento* **44B** 1 (1966).
- [23] L. P. Eisenhart, *Riemannian Geometry*, p. 146 (1949), Princeton.
- [24] S. Engineer, N. Kanekar and T. Padmanabhan, *Mon. Not. R. Astron. Soc.* **314** 279 (2000).